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(DIRAC ALGEBRA) WITH ANY NUMBER OF
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MINIMAL LEFT IDEALS

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INTRODUCTION

IN this note the Clifford-Dirac Algebra generated by n -symbols e_1, e_2, \dots, e_n satisfying the relations

$$e_r e_s + e_s e_r = 2\delta_{rs} \text{ (Kronecker symbol)}$$

over a ground field whose characteristic $\neq 2$ and which contains $\sqrt{-1}$ is resolved into the sum of minimal left ideals. These ideals as well as their bases have been chosen in a suitable manner and the corresponding representation is seen to be identical with the well-known one given by Weyl and Brauer.¹

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§1. It is well known⁺ that the basis elements of the Clifford Algebra with the n symbols e_1, e_2, \dots, e_n satisfying the relation

I. $e_r e_s + e_s e_r = 2\delta_{rs}$ (δ_{rs} Kronecker symbol) are all expressed succinctly by the expression $e_1^{\lambda_1} e_2^{\lambda_2} \dots e_n^{\lambda_n}$ where the λ^s are integres mod 2. Evidently there are 2^n basis elements.

We deduce easily,

II. (1) $e_1 e_2 \dots e_{2r+1}$ commutes with the e_p ; $p \leq 2r + 1$
and
anticommutates with the e_p ; $p > 2r + 1$.

II. (2) $e_1 e_2 \dots e_{2r}$ anticommutates with the e_p ; $p \leq 2r$
and

commutes with the e_p ; $p > 2r$.

$$\text{II. (3) } (e_1 e_2 \dots e_{2r})^2 = (-1)^r$$

$$\text{II. (4) } (e_1 e_2 \dots e_{2r+1})^2 = (-1)^r$$

It follows also that the algebra C_{2n+1} (generated by an odd number of symbols) can be resolved into a direct sum of two simple algebras each of which is isomorphic with a C_{2n} : i.e.,

$$C_{2n+1} = C_{2n+1}\omega + C_{2n+1}(1 - \omega) \text{ where}$$

$$\omega = \frac{1 + e_1 e_2 \dots e_{2n+1}}{2} \text{ if } n \text{ is even or}$$

$$= \frac{1 + i e_1 e_2 \dots e_{2n+1}}{2} \text{ if } n \text{ is odd.}$$

§2. We now take up the complete resolution of a C_{2n} as a direct sum of minimal, mutually orthogonal left ideals. For this purpose we make use of a result due to Witt,² viz., that a Clifford Algebra with $2n$ symbols is the direct product of n such algebras with 2 symbols. Witt has shown that

$$C_{2n} = (1, e_1, e_2) \times (1, i e_1 e_2 e_3, i e_1 e_2 e_4) \times (\dots) \dots \times ($$

where each of the brackets represents an algebra generated by the symbols contained in it. Therefore, the idempotents generating minimal left ideals in C_{2n} are given by

$$\omega_r = \frac{(1 \pm e_1)}{2} \cdot \frac{(1 \pm i e_1 e_2 e_3)}{2} \cdot \dots \cdot \frac{(1 \pm i^{k+1} e_1 e_2 \dots e_{2k-1})}{2} \dots \dots n$$

factors. Corresponding to the two signs \pm in each of the brackets there are evidently 2^n such ω_r 's. We now show that

$$\omega_r^2 = \omega_r \text{ and } \omega_r \omega_s = 0 \text{ if } r \neq s.$$

Now

$$\begin{aligned} \omega_r^2 &= \left\{ \frac{(1 \pm e_1)}{2} \dots \frac{(1 \pm i e_1 e_2 \dots e_{2n-1})}{2} \right\} \\ &\quad \left\{ \frac{(1 \pm e_1)}{2} \dots \frac{(1 \pm i e_1 e_2 \dots e_{2n-1})}{2} \right\} n \text{ even} \\ &= \left(\frac{(1 \pm e_1)}{2} \right)^2 \dots \left(\frac{(1 \pm i e_1 e_2 \dots e_{2n-1})}{2} \right)^2 \\ &= \frac{(1 \pm e_1)}{2} \dots \frac{(1 \pm i e_1 e_2 \dots e_{2n-1})}{2} = \omega_r \end{aligned}$$

To show that $\omega_r \omega_s = 0$ ($r \neq s$) we first observe that in the expressions for ω_r and ω_s , there is at least one bracket which appears with a change of sign in it. Calling this the p th bracket, if it is $\frac{(1 + i e_1 e_2 \dots e_{2p-1})}{2}$ in ω_r , it will be $\frac{1 - i e_1 e_2 \dots e_{2p-1}}{2}$ in ω_s . Since the factors in the brackets commute with one another, we can bring the p th brackets together in $\omega_r \omega_s$. But

$$\frac{(1 + i e_1 e_2 \dots e_{2p-1})}{2} \frac{(1 - i e_1 e_2 \dots e_{2p-1})}{2} = 0$$

Hence

$$\omega_r \omega_s = 0 \text{ if } r \neq s.$$

We next prove that $\sum_{r=1}^{r=2^n} \omega_r = 1$.

Proof.—Let the result be true for $n = m$

$$\text{i.e., } \sum_{r=1}^{r=2^m} \omega_r = \sum_1^{2^m} \frac{(1 \pm e_1)}{2} \dots \frac{(1 \pm i e_1 e_2 \dots e_{2m-1})}{2} = 1 \text{ for } C_{2m}$$

Hence for C_{2m+2} ,

$$\begin{aligned} \sum_1^{2^{m+1}} \omega_r &= \sum_1^{2^{m+1}} \frac{(1 \pm e_1)}{2} \dots \frac{(1 \pm i e_1 \dots e_{2m-1})}{2} \frac{(1 \pm e_1 e_2 \dots e_{2m+1})}{2} \\ &= \left(\sum_1^{2^m} \frac{(1 \pm e_1)}{2} \dots \frac{(1 \pm i e_1 \dots e_{2m-1})}{2} \right) \frac{(1 + e_1 e_2 \dots e_{2m+1})}{2} \\ &\quad + \left(\sum_1^{2^m} \frac{(1 \pm e_1)}{2} \dots \frac{(1 \pm i e_1 \dots e_{2m-1})}{2} \right) \frac{(1 - e_1 e_2 \dots e_{2m+1})}{2} \\ &= \frac{1 + e_1 e_2 \dots e_{2m+1}}{2} + \frac{1 - e_1 e_2 \dots e_{2m+1}}{2} = 1 \end{aligned}$$

i.e., the result is true for $n = m + 1$. But for $n = 1$,

$$\sum \omega_r = \sum \frac{1 \pm e_1}{2} = 1 \text{ and hence it is true for all } n.$$

We now proceed to deduce the irreducible representation of the algebra C_{2^n} by choosing a suitable basis of the minimal left ideal L ; generated by one of ω_r 's, say

$$\omega = \frac{(1+e_1)}{2} \frac{(1+ie_1e_2e_3)}{2} \dots \frac{(1+ie_1e_2\dots e_{2k-1})}{2} \dots \frac{(1+ie_1e_2\dots e_{2n-1})}{2}$$

we first of all show that

$$e_{2k+1} \omega = (-1)^k i e_{2k} \omega \quad k = 1, 2, \dots, n-1.$$

Proof.—(i) Let k be even: From II. 1.

$$\begin{aligned} e_{2k} \omega &= e_{2k} \left(\frac{1+e_1}{2} \right) \dots \left(\frac{1+ie_1e_2\dots e_{2k-1}}{2} \right) \dots \frac{(1+ie_1e_2\dots e_{2n-1})}{2} \\ &= \frac{(1-e_1)}{2} \dots \left(\frac{1-ie_1e_2\dots e_{2k-1}}{2} \right) \left(\frac{e_{2k}-e_1\dots e_{2k-1}e_{2k+1}}{2} \right) \\ &\quad \dots \frac{(1+ie_1\dots e_{2n-1})}{2} \\ e_{2k+1} \omega &= \left(\frac{1-e_1}{2} \right) \dots \left(\frac{1-ie_1\dots e_{2k-1}}{2} \right) \left(\frac{e_{2k+1}+e_1\dots e_{2k}}{2} \right) \\ &\quad \dots \left(\frac{1+ie_1\dots e_{2n-1}}{2} \right) \\ &= \left(\frac{1-e_1}{2} \right) \dots \left(\frac{1-ie_1\dots e_{2k-1}}{2} \right) [e_1\dots e_{2k-1}] \times \\ &\quad \times \left(\frac{e_{2k}-e_1\dots e_{2k-1}e_{2k+1}}{2} \right) \dots \frac{(1+ie_1\dots e_{2n-1})}{2} \text{ using II. 4.} \\ &= \left(\frac{1-e_1}{2} \right) \dots \left(\frac{e_1\dots e_{2k-1}+i}{2} \right) \left(\frac{e_{2k}-e_1\dots e_{2k-1}e_{2k+1}}{2} \right) \\ &\quad \dots \left(\frac{1+ie_1\dots e_{2n-1}}{2} \right) \\ &= i e_{2k} \omega. \end{aligned} \tag{a}$$

(ii) Let k be odd.

$$\begin{aligned} e_{2k} \omega &= \left(\frac{1-e_1}{2} \right) \dots \left(\frac{1-e_1\dots e_{2k-1}}{2} \right) \left(\frac{e_{2k}-ie_1\dots e_{2k-1}e_{2k+1}}{2} \right) \\ &\quad \dots \left(\frac{1+ie_1\dots e_{2n-1}}{2} \right) \end{aligned}$$

$$\begin{aligned}
e_{2k+1} \omega &= \left(\frac{1-e_1}{2} \right) \dots \left(\frac{1-e_1 \dots e_{2k-1}}{2} \right) \left(\frac{e_{2k+1} + i e_1 \dots e_{2k}}{2} \right) \dots \\
&\quad \left(\frac{1 + i e_1 \dots e_{2n-1}}{2} \right) \\
&= \left(\frac{1-e_1}{2} \right) \dots \left(\frac{1-e_1 \dots e_{2k-1}}{2} \right) [i e_1 \dots e_{2k-1}] \times \\
&\quad \times \left(\frac{e_{2k} - i e_1 \dots e_{2k-1} e_{2k+1}}{2} \right) \dots \left(\frac{1 + i e_1 \dots e_{2n-1}}{2} \right) \\
&= \left(\frac{1-e_1}{2} \right) \dots \left(\frac{i e_1 \dots e_{2k-1} - i}{2} \right) \left(\frac{e_{2k} - i e_1 \dots e_{2k-1} e_{2k+1}}{2} \right) \\
&\quad \dots \left(\frac{1 + i e_1 \dots e_{2n-1}}{2} \right) \\
&= -i e_{2k} \omega. \tag{b}
\end{aligned}$$

Combining (a) and (b) we get

$$\underline{e_{2k+1} \omega = (-1)^k i e_{2k} \omega.}$$

We thus see that all the symbols with odd suffixes can be expressed in terms of those with even suffixes only and that $e_1 \omega = \omega$. We therefore take, as the basis elements of the minimal left ideal generated by ω the 2^n terms occurring in

$$\underline{e_{2n}^{\lambda_n} e_{2n-2}^{\lambda_{n-1}} \dots e_2^{\lambda_1} \omega = a_r \omega \text{ where the } \lambda\text{'s are integers mod.2.}}$$

The e 's are written down, as above, with the suffixes, always in the descending order and $\lambda_1, \lambda_2, \dots, \lambda_n$ take the values 0, 1 in the dictionary order. Each a_r represents a particular combination of the e 's and r takes 2^n values. We add a direct proof that the $a_r \omega$ are linearly independent.

Proof.—We first show that if a represents some combination of the e 's, $a\omega_r = \omega_s a$ where ω_r and ω_s are two different mutually orthogonal idempotents. Now an a is of the form

$$a = e_{2l} e_{2m} \dots e_{2s} e_{2t}, \text{ where } n \geq l > m \dots > s > t \geq 1$$

Hence $a\omega_r = e_{2l} e_{2m} \dots e_{2s} e_{2t} \omega_r$.

From II. 1, when e_{2t} is taken to the right of ω_r , one can see easily that there will be a change of sign in the first t brackets only. If now e_{2s} is brought

to the right of ω_r , the signs will be restored in the first t brackets, but a change of sign occurs in the next $s - t$ brackets and the last $n - s$ brackets remain unaltered. We thus observe that when all the e 's are taken to the right of ω_r , it would have changed over to a different orthogonal idempotent ω_s ,

$$i.e., a\omega_r = \omega_s a \quad (r \neq s)$$

Let now $\sum_{r=1}^{r=2^n} a(a_r \omega) = 0$, i.e.,

$$a_1 a_1 \omega + a_2 a_2 \omega + \dots + a_r a_r \omega + \dots + a_{2^n} a_{2^n} \omega = 0$$

$$i.e., a_1 \omega_{s_1} a_1 + a_2 \omega_{s_2} a_2 + \dots + a_r \omega_{s_r} a_r + \dots + a_{2^n} \omega_{s_{2^n}} a_{2^n} \omega = 0$$

Multiply by ω_{s_r} ($r = 1, 2, \dots, 2^n$) on the left. We obtain

$$a_r \omega_{s_r}^2 a = a_r \omega_{s_r} a_r = 0$$

i.e., $a_r = 0$, ($r = 1, 2, \dots, 2^n$) i.e., the $a_r \omega$ are linearly independent.

Choosing these as the basis elements of the left ideal L: generated by ω , we can obtain the matrices of the representation in terms of the Pauli matrices

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The matrices are easily seen to be

$$e_1 \rightarrow P_1 \times P_1 \times P_1 \times \dots \times P_1 \times P_1 \times P_1 \quad n \text{ terms}$$

$$e_{2k} \rightarrow P_1 \times P_1 \times \dots \times P_2 \times P \times E \times E \times \dots \times E \quad ,,$$

$$e_{2k+1} \rightarrow (-1)^k P_1 \times P_1 \times \dots \times P_1 \times P_3 \times E \times \dots \times E_1 \quad ,,$$

where P_2 and P_3 occur in the k th place from the right end in the corresponding expressions.

REFERENCES

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